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On Ground State Solutions of $-\Delta u = u^p - u^q$

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1. INTRODUCTION

In this paper we shall discuss the existence and uniqueness of positive, radially symmetric solutions $u(x)$ of the problem

$$(\mathrm{I})\begin{cases} -\Delta u = u^p - u^q & \text{in } \mathbf{R}^n & (1.1)\\ u > 0 & \text{in } \mathbf{R}^n & (1.2)\\ u(x) \to 0 & \text{as } x \to \infty & (1.3)\end{cases}$$

in which $n > 2$ and

$$q > p > \frac{n+2}{n-2}, \tag{1.4}$$

the last number being the critical Sobolev exponent. If we look for radial

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solutions of (1.1) and (1.2), we would expect u, as a function of the radial variable r, to behave like Ar^{-m}, where A and m satisfy

$$-Am(m+1)r^{-m-2} + A(n-1)mr^{-m-2} = A^p r^{-pm} \tag{1.5}$$

so that *either*

$$m+2 = pm \;\Rightarrow\; m = \frac{2}{p-1} \tag{S}$$

or

$$m+2 < pm, \qquad m+1 = n-1 \;\Rightarrow\; m = n-2. \tag{F}$$

Note that the choice of p relative to the critical Sobolev exponent is such that

$$n-2 > \frac{2}{p-1}$$

so that (F) involves a faster decay than (S). In addition, it follows from (1.5) that for the slower decay, the constant A would be determined, while for the faster decay it would still appear to be free.

In this paper we shall show that there exists precisely one radial solution of Problem (I) with fast decay (F) at infinity.

THEOREM A. *There exists a unique radial solution $u(x)$ of Problem* (I) *such that*

$$u(x) = O(|x|^{-(n-2)}) \qquad as \quad x \to \infty. \tag{1.6}$$

For this solution have

$$\left(\frac{c(p,n)}{c(q,n)}\right)^{1/(q-p)} < u(0) < 1, \tag{1.7}$$

where

$$c(s,n) = \frac{(n-2)s - (n+2)}{2(s+1)}. \tag{1.8}$$

Remark. It follows from Theorem A that there exists a unique constant $A > 0$ such that if $u(r)$ is a solution of Problem (I) which satisfies (1.6), then

$$u(r) \asymp Ar^{-(n-2)} \qquad \text{as} \quad r \to \infty.$$

The motivation for studying ground state solutions of Eq. (1.1) came from the study of the Dirichlet problem

$$(\mathrm{I}_\varepsilon)\begin{cases} -\Delta u = u^p - \varepsilon u^q & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

in which ε is a small nonnegative constant, p and q satisfy (1.4) and B is the unit ball in $\mathbf{R}^n$ [MP]. It is well know that if $\varepsilon = 0$, Problem (I_ε) has no solution. On the other hand, if $\varepsilon > 0$ and small, Problem (I_ε) has at least two ordered solutions. As $\varepsilon \to 0$, the larger of these becomes unbounded at every $x \in B$, while the smaller one, which we denote by u_ε, "concentrates" at the origin, i.e., $u_\varepsilon(0) \to \infty$ and $u_\varepsilon(x) \to 0$ when $x \neq 0$. Near the origin the solution approaches the fast decay ground state U of Problem (I). Specifically, setting $u_\varepsilon(0) = \gamma_\varepsilon$ and $U(0) = c$, one obtains

$$\varepsilon \gamma^{q-p} \to c^{q-p} \qquad \text{as} \quad \varepsilon \to 0$$

and, in terms of the scaled variables

$$\xi = c^{-(p-1)/2} \gamma^{(p-1)/2} x, \qquad v_\varepsilon(\xi) = (c/\gamma) u_\varepsilon(x),$$

one finds that

$$v_\varepsilon(\xi) \to U(\xi) \qquad \text{as} \quad \varepsilon \to 0$$

uniformly in $\mathbf{R}^n$. Thus, in the supercritical case the ground state solution plays the same role as is played in the critical case, when $p = (n+2)/(n-2)$, by the function

$$U_\mu(x) = \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{(n-2)/2}, \qquad \mu > 0,$$

which satisfies the equation

$$-\Delta u = n(n-2) u^{(n+2)/(n-2)}.$$

For further details we refer to [BN, BP, MP] and the literature cited there.

The plan of this paper is the following. In Section 2 we introduce new variables, which transform Eq. (1.1) to a generalized Emden–Fowler equation, and establish the bounds (1.7.) for $u(0)$. Then in Section 3 we give for completeness an existence proof, which is different from that given in [MP]. By far the hardest part, however, is the uniqueness proof which will be given in Sections 4, 5, 6, and 7.

2. Preliminaries

Set

$$t = \left(\frac{n-2}{r}\right)^{n-2}, \qquad y(t) = u(r). \tag{2.1}$$

Then it is a routine calculation to show that (1.1) and (1.2) become

$$y'' + t^{-k} f(y) = 0, \qquad 0 < t < \infty \tag{2.2}$$

$$y > 0, \qquad 0 < t < \infty, \tag{2.3}$$

where

$$f(y) = y^p - y^q \qquad \text{and} \qquad k = \frac{2n-2}{n-2}. \tag{2.4}$$

The boundary conditions on y now become, because $t \to 0$ (∞) corresponds to $r \to \infty$ (0),

$$y(t) \sim \alpha t \qquad \text{as} \quad t \to 0 \tag{2.5}$$

for some $\alpha > 0$ (when we require fast decay as $r \to \infty$) and

$$y(\infty) \text{ finite and } y'(\infty) = 0. \tag{2.6}$$

Clearly, the solution $y(t, \alpha)$ of (2.2) and (2.5) is uniquely determined, because (1.4) implies that $p - k > -1$ and so (2.5) in (2.2) tells us that y'' is integrable down to $t = 0$, and $y'(0) = \alpha$. The question is whether the solution $y(t, \alpha)$ also satisfies (2.6).

THEOREM 1. *There exists a unique choice of* $\alpha > 0$ *such that the solution* $y(t, \alpha)$ *of* (2.2), (2.3), *and* (2.5) *also satisfies* (2.6). *For this solution we have*

$$\left(\frac{c(p, n)}{c(q, n)}\right)^{1/(q-p)} < y(\infty, \alpha) < 1, \tag{2.7}$$

where $c(s, n)$ *has been defined in* (1.8).

Returning to the original variables r and u, Theorem 1 yields Theorem A.

Remark. Although the condition $y'(\infty) = 0$ is natural in view of the condition $u'(0) = 0$, it can be derived from the boundedness of $y(\infty)$. For we have that $k > 2$ and so integration of (2.2) with y bounded implies that $y'(\infty)$ exists, and necessarily $y'(\infty) = 0$. Indeed integration shows that

$$\lim_{t \to \infty} t y'(t) = 0. \tag{2.8}$$

Let $y(t)$ be a solution of (2.2), (2.3), and (2.5). Then, because

$$y'' < 0\ (>0) \qquad \text{when} \quad y < 1\ (>1),$$

$y(t)$ can only be bounded if

$$y(t) < 1, \qquad y'(t) > 0 \qquad \text{for} \quad 0 < t < \infty. \tag{2.9}$$

This means that $y(\infty) \leqslant 1$. However, because $k > 2$, if $y(\infty) = 1$ we would have by uniqueness that $y(t) = 1$ for all $t > 0$ and (2.5) would not be satisfied. Thus for any solution of (2.2), (2.3), (2.5), and (2.6) we must have $y(\infty) < 1$.

To prove the lower bound in (2.7), we multiply (2.2) by y and integrate to obtain

$$yy' + \int_0^t \{-y'^2 + x^{-k} y f(y)\}\, dx = 0. \tag{2.10}$$

If we multiply (2.2) by ty' and integrate we obtain

$$\frac{1}{2} t y'^2 + t^{1-k} F(y) = \int_0^t \left\{ \frac{1}{2} y'^2 + \frac{1-k}{x^k} F(y) \right\} dx, \tag{2.11}$$

where $F(y) = \int_0^y f(s) ds$. If we let $t \to \infty$ in (2.10) and (2.11), and use (2.8) and the fact that $k > 2$, we arrive at

$$\int_0^\infty y'^2\, dx = \int_0^\infty x^{-k} y f(y)\, dx,$$

$$\int_0^\infty y'^2\, dx = 2(k-1) \int_0^\infty x^{-k} F(y)\, dx,$$

and so

$$\int_0^\infty x^{-k} \{ y f(y) - 2(k-1) F(y)\}\, dx = 0.$$

For f defined by (2.4),

$$\begin{aligned} yf(y) - 2(k-1)F(y) &= \frac{p-(2k-3)}{p+1} y^{p+1} - \frac{q-(2k-3)}{q+1} y^{q+1} \\ &= \frac{2}{n-2} \{ c(p,n) y^{p+1} - c(q,n) y^{q+1} \}, \end{aligned}$$

where $c(s, n)$ is defined in (1.8). Hence,

$$c(p, n) \int_0^\infty x^{-k} y^{p+1}\, dx = c(q, n) \int_0^\infty x^{-k} y^{q+1}\, dx$$

$$< c(q, n)\{y(\infty)\}^{q-p} \int_0^\infty x^{-k} y^{p+1}\, dx,$$

because $y(t) < y(\infty)$ for all $t > 0$. Thus

$$\{y(\infty)\}^{q-p} > c(p, n)/c(q, n).$$

3. Existence of a Solution

We prove the existence of a solution by means of a simple shooting argument, which depends upon the following lemmas.

LEMMA 1. *If $\alpha > 0$ is sufficiently large, then the solution y of* (2.2) *and* (2.5) *has the property that y' has a zero strictly before y reaches one.*

LEMMA 2. *If $\alpha > 0$ is sufficiently small, then the solution y of* (2.2) *and* (2.5) *has the property that y reaches the value* 1 *strictly before y' vanishes.*

Accepting these lemmas for the moment, we consider the sets S^+ and S^- defined by

$$S^+ = \{\alpha > 0 : y' = 0 \text{ strictly before } y = 1\},$$

$$S^- = \{\alpha > 0 : y = 1 \text{ strictly before } y' = 0\}.$$

Clearly, S^+ and S^- are disjoint, they are nonempty by Lemmas 1 and 2, and they are easily seen to be open as a consequence of the continuous dependence on initial data. Indeed, if $\alpha_1 \in S^+$, then there exists a $t_1 > 0$ such that

$$y'(t_1, \alpha_1) = 0, \qquad y(t, \alpha_1) < 1 \qquad \text{for} \quad 0 < t \leqslant t_1.$$

Since $y''(t_1, \alpha_1) < 0$ it follows that $y'(t, \alpha)$ has a zero close to t_1 if α is sufficiently close to α_1 and we will continue to have $y < 1$. Therefore, $\alpha \in S^+$ for α sufficiently close to α_1.

Thus there exists some α, say α_0, in neither S^+ nor S^-. Clearly we must have that *either* $y' = 0$ and $y = 1$ simultaneously, *or* $y' > 0$ and $y < 1$ for all $t \geqslant 0$. Because the first possibility would imply that $y(t, \alpha_0) = 1$ for all $t > 0$, the second possibility must hold. This completes the proof of the existence of a solution of (2.2), (2.3), (2.5), and (2.6).

It remains to prove the two lemmas.

Proof of Lemma 1. We make the scaling,

$$t = \alpha^{-L} T, \qquad y(t) = \alpha^{-M} Y(T), \tag{3.1}$$

where

$$L = \frac{p-1}{p-k+1}, \qquad M = \frac{k-2}{p-k+1}. \tag{3.2}$$

It is easy to verify that (2.2) then becomes

$$Y'' + T^{-k}(Y^p - \alpha^{-\mu} Y^q) = 0, \tag{3.3}$$

where

$$\mu = \frac{(k-2)(q-p)}{p-k+1}, \tag{3.4}$$

and (2.5) becomes

$$Y(0) = 0, \qquad Y'(0) = 1. \tag{3.5}$$

Because $\mu > 0$, we see that the solution Y of (3.3) and (3.5) behaves for large α on compact T-intervals like the solution Y_0 of the problem

$$Y'' + T^{-k} Y^p = 0, \qquad T > 0 \tag{3.6}$$

$$Y(0) = 1, \qquad Y'(0) = 1. \tag{3.7}$$

We shall show below that Y_0 has the property that $Y_0' = 0$ at some finite value T_0 of T. Clearly $Y_0(T_0)$ is finite and does not depend on α. From the scaling we then deduce that $y'(\alpha^{-L} T_0) = 0$ and that $y(\alpha^{-L} T_0)$ is small, so that the lemma is proved.

Suppose to the contrary that

$$Y(T) > 0, \qquad Y'(T) > 0, \qquad Y''(T) < 0 \qquad \text{for all} \quad T > 0,$$

where we have dropped the subscript zero for convenience. Then

$$Y(T) = \int_0^T Y'(x)\, dx = TY'(T) - \int_0^T xY''(x)\, dx > TY'(T).$$

Hence, by (3.6),

$$Y'' < -T^{p-k}(Y')^p$$

and on integration over $[T_1, T_2]$,

$$\left[\frac{1}{1-p}(Y')^{1-p}\right]_{T_1}^{T_2} < -\left[\frac{1}{p-k+1} T^{p-k+1}\right]_{T_1}^{T_2}.$$

If we now fix T_1 and let T_2 become large, we see that

$$Y'(T) \to 0 \qquad \text{as} \quad T \to \infty$$

because $p - k + 1 > 0$. Indeed we have

$$(Y')^{1-p} > \frac{p-1}{p-k+1} T^{p-k+1}.$$

Thus, with K not necessarily the same constant at each appearance, we obtain

$$Y'(T) < KT^{\{(k-2)/(p-1)\}-1},$$

and so upon integration,

$$Y(T) < KT^{(k-2)/(p-1)}.$$

Therefore,

$$TY'^2 < KT^\nu \qquad \text{and} \qquad YY' < KT^\nu,$$

where

$$\nu = -1 + 2\frac{k-2}{p-1} = \frac{2k-3-p}{p-1} < 0,$$

and so

$$TY'^2,\ YY' \to 0 \qquad \text{as} \quad T \to \infty.$$

We can now repeat the calculations of Section 2 with T and Y instead of t ad y, and $f(Y) = Y^p$. Letting $T \to \infty$ in (2.10) and (2.11) we obtain

$$\left(1 - 2\frac{k-1}{p+1}\right) \int_0^\infty x^{-k} y^{p+1}(x)\, dx = 0,$$

a contradiction.

Remark. The above result can also be proved using the classical Fowler transformation and an energy argument [HH].

Proof of Lemma 2. Suppose for contradiction that with α small, $y < 1$ as long as $y' > 0$. Then, if

$$t_1 = \sup\{t > 0 : y' > 0 \text{ on } (0, t)\}$$

we have

$$y'(t) < \alpha \quad \text{and} \quad y(t) < \alpha t, \quad 0 < t < t_1.$$

Substitution into the differential equation yields

$$y''(t) > -\alpha^p t^{p-k}, \quad 0 < t < t_1$$

and after integration

$$y'(t) > \alpha - \frac{\alpha^p}{p-k+1} t^{p-k+1}, \quad 0 < t < t_1 \tag{3.8}$$

$$y(t) > \alpha t - \frac{\alpha^p}{(p-k+1)(p-k+2)} t^{p-k+2}, \quad 0 < t < t_1. \tag{3.9}$$

From (3.8) we deduce that $t_1 > t_0$, where t_0 is determined by

$$\alpha - \frac{\alpha^p}{p-k+1} t_0^{p-k+1} = 0.$$

At t_0 we have according to (3.9),

$$\begin{aligned} y(t_0) &> t_0 \left(\alpha - \frac{\alpha^p}{(p-k+1)(p-k+2)} t_0^{p-k+2} \right) \\ &= t_0 \left(\alpha - \frac{\alpha}{p-k+2} \right) \\ &= K\alpha^{-(k-2)/(p-k+1)}. \end{aligned}$$

Since $p-k+1>0$ we will have that $y(t_0) > 1$ if α is small enough, yielding a contradiction.

4. Uniqueness of the Solution

The remainder of the paper is devoted to proving that the solution is unique. For this we follow the method of Kwong [K], and define the function $w = \partial y/\partial \alpha$. It is readily seen to be a solution of the problem

$$w'' + t^{-k}(py^{p-1} - qy^{q-1})w = 0, \quad t > 0 \tag{4.1}$$

$$w(0) = 0, \quad w'(0) = 1. \tag{4.2}$$

As with y, we shall denote the dependence of w on α by $w(t, \alpha)$.

The proof of uniqueness relies on the following two technical lemmas.

LEMMA 3. *Let $\alpha > 0$ be such that $y' = 0$ first at some point $\bar{t} > 0$. Then there exists a point $\tau_1 \in (0, \bar{t})$ so that*

$$w(\tau_1) = 0, \qquad w(t) < 0 \qquad \textit{for} \quad \tau_1 < t \leqslant \bar{t} \tag{4.3}$$

and

$$w'(\bar{t}) < 0.$$

LEMMA 4. *Let $\alpha_0 > 0$ be such that*

$$y'(t, \alpha_0) > 0, \qquad y(t, \alpha_0) < 1 \qquad \textit{for all} \quad t \geqslant 0.$$

If $\alpha \geqslant \alpha_0$, and *$\alpha - \alpha_0 < \eta$ for some sufficiently small η, then there exist values $T > 0$ and $\delta > 0$, which depend on η but not otherwise on α, such that*

(a) *$y'(t, \alpha) > 0$ for $0 \leqslant t \leqslant T$;*

(b) *$w(\tau_1, \alpha) = 0$ at some first $\tau_1 \in (0, T)$ and $w(t, \alpha) < 0$ for $\tau_1 < t \leqslant T$;*

(c) *$w'(t, \alpha) \leqslant -\delta$ for $t \geqslant T$ as long as $y' > 0$ and $y < 1$.*

Before proving Lemmas 3 and 4 we show how they are used to obtain the uniqueness result.

Choose $\eta > 0$, $T > 0$, and $\delta > 0$ such that Lemma 4 holds. Next, let $\alpha \in (\alpha_0, \alpha_0 + \eta)$ be arbitrarily chosen and fixed. Then, $y'(t, \alpha) > 0$ for $0 \leqslant t \leqslant T$. By the mean value theorem and Lemmas 3 and 4, we see that for $t \geqslant T$, as long as $y'(t, \alpha) > 0$ and $y(t, \alpha) < 1$, we have

$$y'(t, \alpha) - y'(t, \alpha_0) = w'(t, \tilde{\alpha})(\alpha - \alpha_0) \leqslant -\delta(\alpha - \alpha_0) \tag{4.4}$$

for some $\tilde{\alpha} \in (\alpha_0, \alpha)$. Since $y'(t, \alpha_0) \to 0$ as $t \to \infty$ we conclude that $y'(\bar{t}, \alpha) = 0$ at some first $\bar{t} > T$. Thus, if $\alpha \in (\alpha_0, \alpha_0 + \eta)$, then $y'(t, \alpha)$ has a first positive zero, say at $t = \bar{t}(\alpha)$, and so as noted earlier

$$y''(\bar{t}(\alpha), \alpha) < 0 \qquad \text{and} \qquad y(\bar{t}(\alpha), \alpha) < 1. \tag{4.5}$$

We differentiate the equation

$$y'(\bar{t}(\alpha), \alpha) = 0$$

with respect to α and obtain

$$y''(\bar{t}) \frac{d\bar{t}}{d\alpha} + w'(\bar{t}) = 0. \tag{4.6}$$

It follows from Lemma 3 that $w'(\bar{t})<0$. Therefore, by the first inequality in (4.5)

$$\frac{d\bar{t}}{d\alpha}<0 \qquad \text{for} \quad \alpha_0<\alpha<\alpha_0+\varepsilon.$$

If there were a first $\bar{\alpha}>\alpha_0$ for which $d\bar{t}/d\alpha=0$, then (4.6) would imply that $w'(\bar{t}(\bar{\alpha}), \bar{\alpha})=0$, which contradicts Lemma 3. Therefore,

$$\frac{d\bar{t}}{d\alpha}<0 \qquad \text{for all} \quad \alpha>\alpha_0.$$

From this it follows that $y'(t, \alpha)$ has a first positive zero for each $\alpha>\alpha_0$ and so there cannot exist an $\alpha_1>\alpha_0$ for which

$$y'(t, \alpha_1)>0 \qquad \text{and} \qquad 0<y(t, \alpha_1)<1 \qquad \text{for all} \quad t>0.$$

This completes the proof of the uniqueness theorem.

It remains to prove Lemmas 3 and 4. We do this in the succeeding sections.

5. Three Preliminary Lemmas

Proceeding as in Kwong [K], we develop a comparison function

$$v=ty'+\beta y$$

for an appropriately chosen value of β. The function v satisfies

$$v''+t^{-k}(py^{p-1}-qy^{q-1})v=t^{-k}\phi \tag{5.1}$$

$$v(0)=0, \qquad v'(0)=(\beta+1)\alpha, \tag{5.2}$$

where

$$\phi=(k-2-\beta)(y^p-y^q)+\beta(py^p-qy^q). \tag{5.3}$$

The technical properties of v which we shall need are given in the next lemma.

Lemma 5. *Let $\alpha>0$ be given, and let y be the solution of* (2.2) *and* (2.5). *Then there exist values $\beta(\alpha)$ and $\rho(\alpha)$ such that*

(a) $y'>0$ *in* $[0, \rho]$;

(b) $v>0$, $\phi<0$ *in* $(0, \rho)$;

(c) $v(\rho)=\phi(\rho)=0$, $v'(\rho)<0$;

(d) $\phi(t)>0$ *for* $t>\rho$ *so long as* $y'(t)>0$;

(e) $-1<\beta<-(k-2)/(p-1)$.

Further, given a particular set of values α, k, p, and q satisfying

$$\alpha > 0, \qquad k > 2, \qquad q > p > (n+2)/(n-2),$$

and a corresponding pair (β, ρ), then we can continue (β, ρ) uniquely as continuous functions of α, k, p, and q so that (a)–(e) *are still satisfied.*

Proof. The statement $\phi = 0$ is equivalent to

$$y^{q-p} = \frac{k-2+\beta(p-1)}{k-2+\beta(q-1)} \stackrel{\text{def}}{=} N, \tag{5.4}$$

and $\phi < 0$ (>0) if $y^{q-p} < N$ ($>N$). When β, k, p, and q are such that (5.4) has a solution $y = y(t)$ for some t, then we define t_2 to be the smallest such value of t. Thus, if there exists such a value of t, then

$$\phi(t_2) = 0 \qquad \text{and} \qquad y'(t) > 0 \qquad \text{for} \quad 0 \leqslant t < t_2. \tag{5.5}$$

If there exists no such value, then either the solution y has a first maximum less than $N^{1/(q-p)}$, in which case we define t_2 to be the location of this maximum, or y increases to a limit at ∞ with $y(\infty) \leqslant N^{1/(q-p)}$, in which case we define $t_2 = \infty$. In all cases we have

$$y'(t) > 0 \qquad \text{for} \quad 0 \leqslant t < t_2. \tag{5.6}$$

If $\beta = -1$, then v is initially negative, because in that case

$$v(t) = ty'(t) - y(t) < 0$$

by the concavity of y. If β is just slightly greater than -1, then v is initially positive because

$$v' = ty'' + (\beta + 1)y',$$

but by continuity in β, it must become negative at small values of t. For each $\beta > -1$ we define t_1 to be the first positive zero of v. Then we may certainly suppose that

$$y' > 0 \qquad \text{in } [0, t_1]. \tag{5.7}$$

By definition we have

$$v > 0 \qquad \text{in } (0, t_1) \qquad \text{and} \qquad v(t_1) = 0, \tag{5.8}$$

and we may also suppose that

$$v'(t_1) < 0. \tag{5.9}$$

For at t_1, v'' and ϕ have by (5.1) the same sign and when β is close to -1, $y(t_1)$ is small so that $\phi(t_1)$ has the same sign as $k-2+\beta(p-1)$, which is negative when β is close to -1. Thus, if $v'(t_1)=0$, v would have a maximum at t_1, which is impossible. Finally we note that we will also have

$$t_2 > t_1. \tag{5.10}$$

We now increase β and follow t_2 and t_1. If t_2 is defined by (5.5), then because $\partial N/\partial\beta < 0$ and $y' > 0$, t_2 decreases. On the other hand, if t_2 is not defined by (5.5) it is independent of β. Thus we can assert that t_2 always moves continuously to the left if it moves at all (with an obvious interpretation if $t_2 = \infty$).

Turning to t_1, we note that as long as (5.10) holds, and so $\phi < 0$ on $(0, t_1)$, v cannot collect a zero on $(0, t_1)$. This would imply that at some $t^* < t_1$,

$$v(t^*)=0, \qquad v'(t^*)=0$$

and then, since $\phi(t^*)<0$, (5.1) would tell us that v has a maximum at t^* which is not possible. By the implicit function theorem and (5.9) applied (in obvious notation) to the equation

$$v(t_1(\beta), \beta)=0,$$

we see that t_1 is a differentiable function of β and moves continuously to the right. Thus, so long as (5.10) holds, we also have (5.8) and (5.9) holding, and it is a question of whether (5.7) or (5.10) fails first.

Since $t_2=0$ would correspond to $\beta = -(k-2)/(p-1)$ and $t_1>0$, it follows that if the inequality (5.10) breaks, it does so for $\beta < -(k-2)/(p-1)$.

If the inequality (5.7) were to fail first, it cannot fail at an interior point of $[0, t_1]$, because then we would have $y'=y''=0$ at that point and hence $y \equiv 1$. If (5.7) were to break at t_1, then we would have $v(t_1)=0$ and $y'(t_1)=0$ and so $\beta=0$. This proves that (5.10) will fail first.

Furthermore, when (5.10) fails, the point t_2 must be defined as in (5.5). For if $t_1=t_2$ were finite and such that $y'(t_2)=0$, then $\beta=0$, contradicting $\beta < -(k-2)/(p-1)$. And if $t_1=t_2=\infty$, then $y(\infty)$ must be finite and, by (2.8), $ty'(t) \to 0$ as $t \to \infty$. So again, we have $\beta=0$.

The lemma is now proved except for part (d), which is obvious, and the continuous dependence of (β, ρ) on α, k, p, and q, Since (β, ρ) can be determined by solving (in an obvious notation)

$$v(\rho, \beta)=0, \qquad \phi(\rho, \beta)=0,$$

the continuous dependence will be proved if we can show that

$$\frac{\partial(v, \phi)}{\partial(\rho, \beta)} \neq 0.$$

In fact

$$\frac{\partial(v, \phi)}{\partial(\rho, \beta)} = \begin{vmatrix} v'(\rho) & \phi'(\rho) \\ y(\rho) & (p-1)\, y^p - (q-1)\, y^q \end{vmatrix}.$$

Now we have

$$v'(\rho) < 0, \qquad \phi'(\rho) \geqslant 0, \qquad y(\rho) > 0.$$

Also, since $\phi(\rho) = 0$, we can write

$$(p-1)\, y^p - (q-1)\, y^q = -\frac{k-2}{\beta}\,(y^p - y^q). \tag{5.11}$$

Because $p < q$ implies that $N < 1$ and so by (5.5), $y < 1$, we see that both sides of (5.11) are strictly positive. Hence, the Jacobian is strictly negative and the lemma is proved.

LEMMA 6. *There exists a set of values α, k, p, and q such that the corresponding solution y has the following properties*:

(a) *there exists a finite t_0 which is the first zero of y'*;

(b) *the function*

$$\theta(t) = \frac{t y'(t)}{y(t)}$$

satisfies

$$\theta'(t) < 0 \quad \text{for} \quad 0 < t < t_0, \qquad \theta(t_0) = 0;$$

(c) *if β and ρ are chosen as in Lemma 5, then*

$$v(t) < 0 \quad \text{for} \quad \rho < t \leqslant t_0.$$

Proof. We take α to be large and make the scaling used in the proof of Lemma 1, noting that it does not alter the function θ. Lemma 1 itself then assures us that part (a) is true and we require only to prove parts (b) and (c).

In view of the convergence of the scaled solution Y, for large α and on compact sets in T, to the solution Y_0 of the problem

$$Y'' + T^{-k} Y^p = 0, \qquad t > 0$$
$$Y(0) = 0, \qquad Y'(0) = 1,$$

we see that part (b) is proved if we can find k, p, and q such that the same holds for the function

$$\theta_0(T) = \frac{T Y_0'(T)}{Y_0(T)}.$$

We may readily verify that

$$\theta_0' = T^{-1}\theta_0(1-\theta_0) - T^{1-k} Y_0^{p-1}$$

or, with $s = \log T$,

$$\frac{d\theta_0}{ds} = \theta_0(1-\theta_0) - e^{-(k-2)s} Y_0^{p-1} \tag{5.12}$$

and

$$\frac{d^2\theta_0}{ds^2} = (1-2\theta_0)\frac{d\theta_0}{ds} - e^{-(k-2)s} Y_0^{p-1}\{-(k-2) + (p-1)\,\theta_0\}. \tag{5.13}$$

Using (5.12) to eliminate Y_0 from (5.13), we finally obtain

$$\begin{aligned} \frac{d^2\theta_0}{ds^2} = \frac{d\theta_0}{ds}&\{1 - 2\theta_0 + (p-1)\,\theta_0 - (k-2)\} \\ &- \theta_0(1-\theta_0)\{(p-1)\,\theta_0 - (k-2)\}. \end{aligned} \tag{5.14}$$

We shall make the choice, as we may, that k and p are such that

$$\gamma \equiv \frac{k-2}{p-1} < \frac{1}{2}. \tag{5.15}$$

At $T = 0$ or $s = -\infty$, we have $\theta_0 = 1$, and from (5.13) in the form

$$\frac{d}{ds}\left(A(s)\frac{d\theta_0}{ds}\right) = -(p-1)\, Y_0^{p-1} e^{-(k-2)s} A(s)(\theta_0 - \gamma),$$

where

$$A(s) = \exp\left(-\int_0^s [1 - 2\theta_0(u)]\, du\right),$$

we see that

$$A(s)\frac{d\theta_0}{ds} < 0$$

and so

$$\frac{d\theta_0}{ds} < 0$$

at least until θ_0 decreases to the value γ, and in particular until $\theta_0 = \frac{1}{2}$. But (5.13) tells us that

$$\frac{d\theta_0}{ds} - \theta_0(1-\theta_0)$$

is decreasing until $\theta_0 = \gamma$, and so

$$\left\{\frac{d\theta_0}{ds} - \theta_0(1-\theta_0)\right\}\bigg|_{\theta_0=\gamma} < \left\{\frac{d\theta_0}{ds} - \theta_0(1-\theta_0)\right\}\bigg|_{\theta_0=1/2}$$

so that

$$\frac{d\theta_0}{ds}\bigg|_{\theta_0=\gamma} < \gamma(1-\gamma) - \frac{1}{4}.$$

Now consider what happens as θ_0 decreases below γ. So long as $\theta_0'' < 0$ (and certainly $\theta_0'' < 0$ at $\theta_0 = \gamma$), we see that θ_0' increases in modulus and so in modulus exceeds $\frac{1}{4} - \gamma(1-\gamma)$. Thus the first term in (5.14) exceeds in modulus

$$\{\tfrac{1}{4} - \gamma(1-\gamma)\}\{1 - 2\gamma - (k-2)\}, \tag{5.16}$$

while the second term does not exceed

$$(k-2)\,\gamma(1-\gamma). \tag{5.17}$$

We have so far insisted that $\gamma < \frac{1}{2}$. By picking a value of k sufficiently close to 2 and thus γ sufficiently small, we can clearly arrange that (5.16) exceeds (5.17) and then $\theta_0'' < 0$ and $\theta_0' < 0$ until $\theta_0 = 0$. Since $\theta = 0$ corresponds to $y' = 0$, part (b) of the lemma is proved.

To prove part (c), we note that

$$\theta' < 0 \qquad \text{in } (0, t_0),$$

and since $\theta(\rho) = -\beta$ it follows that

$$\theta < -\beta \qquad \text{in } (\rho, t_0].$$

Hence,

$$v < 0 \qquad \text{in } (\rho, t_0].$$

This completes the proof of the lemma.

We now want to extend the result of Lemma 6 to all α, k, p, and q by continuity. This result is contained in the following lemma.

LEMMA 7. *Let* $n > 2$, $k = (2n-2)/(n-2)$, $q > p > (n+2)/(n-2)$ *and* $\alpha > 0$. *Then there exist values* β *and* ρ *which satisfy the conditions of Lemma* 5, *obtained as continuous functions of* k, p, q, *and* α *from the specific values in Lemma* 6, *such that*

(a) $v(t) < 0$ *for* $t > \rho$, *so long as* $y'(t) > 0$;

(b) *if* $y'(t) = 0$ *for some* $t > \rho$, *then* $v(t) < 0$;

(c) *if* $y'(t) > 0$ *for all* $t > 0$ *and* $y(\infty) < 1$, *then* $v(\infty) = \beta y(\infty) < 0$.

Proof. Given any point $P = (k, p, q, \alpha)$ we connect it by a continuous path in $\mathbf{R}^4$, with k, p, q, α always satisfying the relevant inequalities, to the point $P_0 = (k_0, p_0, q_0, \alpha_0)$ of Lemma 6. For P sufficiently close to P_0, Lemma 5 assures us that we can define (β, ρ) uniquely as continuous functions of P so that the conditions of Lemma 5 continue to hold, and this process can be repeated unless we reach a point P_1, say, where the conditions of Lemma 5 cease to hold.

We now consider what happens as $P \to P_1$. We have $\beta \to \beta_1$ and $\rho \to \rho_1$, say, at least by some subsequence, where

$$-1 \leqslant \beta_1 \leqslant -(k-2)/(p-1)$$

and, possibly, $\rho_1 = \infty$. In fact, $\beta_1 = -1$ would imply v initially negative, and so $\rho_1 = 0$. But, by the definition of ϕ, $\rho_1 = 0$ implies that $\phi > 0$ for t just greater than ρ_1, a situation which cannot arise as a limit of the situation in Lemma 5. Similarly, $\beta = -(k-2)/(p-1)$ also implies, by (5.4), that $\rho_1 = 0$. So

$$-1 < \beta_1 < -(k-2)/(p-1).$$

If $\rho_1 = \infty$, we would have $y' > 0$ on $(0, \infty)$ and $y(\infty) \leqslant N^{1/(q-p)}$ from (5.4). Remembering (2.8) we conclude that $v(\infty) = \beta y(\infty) < 0$. Hence, ρ_1 must be finite.

With $\beta = \beta_1$, $\rho = \rho_1$ condition (c) of Lemma 5 is clearly satisfied, and (b) is satisfied because ρ_1 is the first zero of ϕ (from (5.4)), and v cannot have an internal zero in $(0, \rho_1)$ by the type of argument we have used before. Also, (a) is satisfied because y' cannot have an internal zero. Part (d) of Lemma 5 is obvious.

The continuation of (β, ρ) as continuous functions of P satisfying the conditions of Lemma 5 is thus established, and it is mainly a matter of showing that in this continuation process the conditions (a)–(c) of the present lemma also continue to hold.

The condition (a) follows because v cannot have an internal zero in (ρ, t_1) if t_1 is the first zero of y'. For $v = v' = 0$ implies that $v'' > 0$ since $\phi > 0$, and this would imply a minimum for v which is absurd. Also, $v(t_1) = 0$ implies $\beta = 0$, which is also absurd. This proves (b). Finally (c) follows because then $ty' \to 0$ as $t \to \infty$.

This completes the proof of Lemma 7.

6. Proof of Lemma 3

Let

$$h = ty'.$$

Then

$$h'' + \frac{1}{t^k}(py^{p-1} - qy^{q-1})h = \frac{k-2}{t^k}(y^p - y^q) \tag{6.1}$$

with

$$h(0) = 0, \qquad h'(0) = \alpha > 0. \tag{6.2}$$

We introduce the Wronskian

$$H = wh' - hw', \tag{6.3}$$

which satisfies

$$H' = \frac{k-2}{t^k}(y^p - y^q)\, w \tag{6.4}$$

and

$$H(0) = 0, \qquad H'(0) = 0. \tag{6.5}$$

We also consider the Wronskian

$$G = wv' - vw' \tag{6.6}$$

which satisfies

$$G' = \frac{\phi w}{t^k} \tag{6.7}$$

and

$$G(0) = 0, \qquad G'(0) = 0. \tag{6.8}$$

Now we suppose for contradiction that $w > 0$ in $(0, \bar{t})$, $\bar{t}$ being the first zero of y'. Then $H' > 0$ in $(0, \bar{t})$ and so $H(\bar{t}) > 0$. But

$$H(\bar{t}) = w(\bar{t})\, h'(\bar{t}) - h(\bar{t})\, w'(\bar{t}) = w(\bar{t})\, \bar{t}y''(\bar{t}) \leqslant 0.$$

giving the required contradiction.

The definition fo τ_1 as the first zero of w implies that $w'(\tau_1) \leqslant 0$, and in fact $w'(\tau_1) < 0$ since otherwise $w \equiv 0$.

We now assert that ρ, as in Lemma 5, is such that $\rho < \tau_1$. For if we suppose for contradiction that $\rho \geqslant \tau_1$, then $\phi < 0$ on $(0, \tau_1)$ and so $G' < 0$ on $(0, \tau_1)$ which means that $G(\tau_1) < 0$. But

$$G(\tau_1) = -v(\tau_1)\, w'(\tau_1) \geqslant 0$$

giving the necessary contradiction.

Thus $0 < \rho < \tau_1$, and it follows from Lemmas 5 and 7 that $v < 0$ and $\phi > 0$ on $(\rho, \bar{t})$. Suppose for contradiction that there exists a $\tau_2 \in (\tau_1, \bar{t}]$ such that $w(\tau_2) = 0$ and $w < 0$ on (τ_1, τ_2). Then $G' < 0$ on (τ_1, τ_2), so that $G(\tau_1) > G(\tau_2)$. On the other hand, $G(\tau_1) \leqslant 0$ and $G(\tau_2) \geqslant 0$, giving the required contradiction.

It now remains to prove that $w'(\bar{t}) < 0$. We know now that $G' < 0$ on $(\tau_1, \bar{t})$ so that $G(\bar{t}) < 0$. But

$$G(\bar{t}) = w(\bar{t})\, v'(\bar{t}) - v(\bar{t})\, w'(\bar{t})$$

and

$$w(\bar{t}) < 0, \qquad v'(\bar{t}) = \bar{t}y''(\bar{t}) < 0, \qquad v(\bar{t}) < 0$$

so that necessarily $w'(\bar{t}) < 0$.

7. Proof of Lemma 4

We first examine the situation at $\alpha = \alpha_0$. By Lemmas 3 and 5 there exist values $\tau_1 > \rho > 0$ such that

$$v(\rho) = \phi(\rho) = 0, \qquad v < 0, \quad \phi > 0 \quad \text{in } (\rho, \infty),$$

$$w(\tau_1) = 0, \qquad w < 0 \quad \text{in } (\tau_1, \infty).$$

Further w'' must ultimately be of one sign, since w is, and y is monotonic. Therefore, $w'(\infty)$ exists. Clearly $w'(\infty) \leqslant 0$, and possibly $w'(\infty) = -\infty$. In fact $w'(\infty) = 0$ is impossible since, if so, then $w(t) = o(t)$ as $t \to \infty$ and

$$wv' = w\{ty'' + (\beta + 1)\, y'\} \to 0 \qquad \text{as} \quad t \to \infty,$$

and, because $v(\infty)$ exists, $vw' \to 0$ as $t \to \infty$ as well. Hence, in that case, $G(t) \to 0$ as $t \to \infty$. But $G' < 0$ in (τ_1, ∞) so that $G(\infty) < G(\tau_1) \leqslant 0$, giving a contradiction.

To determine the behaviour for α sufficiently close to α_0 we consider two cases.

(i) $w'(\infty, \alpha_0) = -\infty$.

In this case $w''(t, \alpha_0) < 0$ for sufficiently large t. Thus there exist values $\delta > 0$ and $T > \tau_1$ such that

$$w(t, \alpha_0) < -\delta, \qquad w'(t, \alpha_0) < -\delta, \qquad y(t, \alpha_0) > \left(\frac{p}{q}\right)^{1/(q-p)}$$

for all $t \geqslant T$. With δ and T thus fixed and $\alpha - \alpha_0$ sufficiently small, we have

$$w(T, \alpha) < -\delta, \qquad w'(T, \alpha) < -\delta, \qquad y(T, \alpha) > \left(\frac{p}{q}\right)^{1/(q-p)}, \qquad y'(T, \alpha) > 0,$$

and from (4.1) we see that these inequalities continue to hold, with t in place of T, until $y'(t) = 0$. This proves part (c) of the lemma in this case, and parts (a) and (b) are easy.

(ii) $w'(\infty, \alpha_0) = -\eta$, η positive and finite.

We define

$$M = \sup\{|py^{p-1} - qy^{q-1}| : 0 \leqslant y \leqslant 1\}.$$

Since $w'(t) \to -\eta$ as $t \to \infty$, there exists a T such that

$$-\tfrac{4}{3}\eta T < w(T, \alpha_0) < -\tfrac{3}{4}\eta T,$$
$$-\tfrac{4}{3}\eta < w'(T, \alpha_0) < -\tfrac{3}{4}\eta,$$

and

$$\left(\frac{4}{3(k-1)} + \frac{3}{2(k-2)}\right) MT^{2-k} < \frac{1}{6}. \tag{7.1}$$

With T thus fixed, we see that, if $\alpha - \alpha_0$ is sufficiently small, then

$$y'(t, \alpha) > 0, \qquad y(t, \alpha) < 1 \qquad \text{for} \quad t \in [0, T],$$
$$-\tfrac{4}{3}\eta T < w(T, \alpha) < -\tfrac{3}{4}\eta T,$$
$$-\tfrac{4}{3}\eta < w'(T, \alpha) < -\tfrac{3}{4}\eta.$$

Let

$$T_\alpha = \sup\{t > T : y' > 0 \text{ and } y < 1 \text{ in } (T, t)\}.$$

If $t \in (T, T_\alpha)$ and if $-\frac{3}{2}\eta < w' < 0$, then

$$w(t) \geqslant -\tfrac{4}{3}\eta T - \tfrac{3}{2}\eta(t - T) \tag{7.2}$$

and so, from (4.1),

$$w''(t) \geqslant -\tfrac{4}{3}\eta TMt^{-k} - \tfrac{3}{2}\eta Mt^{1-k}$$

and

$$\begin{aligned} w'(t) &> -\frac{4}{3}\eta + \frac{4}{3}\frac{\eta TM}{k-1}(t^{1-k} - T^{1-k}) + \frac{3}{2}\frac{\eta M}{k-2}(t^{2-k} - T^{2-k}) \\ &> -\frac{4}{3}\eta - \left(\frac{4}{3(k-1)} + \frac{3}{2(k-2)}\right)\eta MT^{2-k}. \end{aligned}$$

From (7.1) it now follows that

$$w' > -\tfrac{3}{2}\eta \qquad \text{in} \quad (T, T_\alpha)$$

as long as $w' < 0$.

But from (7.2) and (4.1) we also have

$$w''(t) \leqslant \tfrac{4}{3}\eta TMt^{-k} + \tfrac{3}{2}\eta Mt^{1-k}$$

and

$$\begin{aligned} w'(t) &< -\frac{3}{4}\eta - \frac{4}{3}\frac{\eta TM}{k-1}(t^{1-k} - T^{1-k}) - \frac{3}{2}\frac{\eta M}{k-2}(t^{2-k} - T^{2-k}) \\ &< -\frac{3}{4}\eta + \left(\frac{4}{3(k-1)} + \frac{3}{2(k-2)}\right)\eta MT^{2-k} \\ &< -\frac{7}{12}\eta \end{aligned}$$

for all $t \in (T, T_\alpha)$ because of (7.1).

This completes the proof.

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